

# Active to absorbing state phase transition in the presence of a fluctuating environment: Feedback and universality

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We construct and analyse a simple *reduced* model to study the effects of the interplay between a density undergoing an active-to-absorbing state phase transition (AAPT) and a fluctuating environment in the form of a broken symmetry mode coupled to the density field in any arbitrary dimension. We show, by using perturbative renormalisation group calculations, that *both* the effects of the environment on the density and the latter's feedback on the environment influence the ensuing universal scaling behaviour of the AAPT at its extinction transition. Phenomenological implications of our results in the context of more realistic natural examples are discussed.

## I. INTRODUCTION

The phenomena of active to absorbing state phase transition (AAPT) forms a paradigmatic example of non-equilibrium phase transitions. The enumeration of the scaling exponents that characterise the AAPT and the corresponding universality classes are topics of intense research activities at present [1]. It is now generally believed, as enunciated in what is known as the *Directed Percolation Hypothesis* [2], that in the absence of any special symmetry, conservation law, quenched disorder or long-ranged interactions the AAPT belongs to the directed percolation (DP) universality class, as long as there is a single absorbing state. Well-known examples of models belonging to the DP universality class include the Gribov [3] process or the epidemic process with recovery and the stochastic formulations of the predator prey automaton models [1]. Continuum versions of models belonging to the DP universality class are described formally by the Reggeon field theory [4–6], which is a stochastic multiparticle process that describes the essential features of local growth processes of populations in a uniform environment near their extinction threshold [7, 8]. The model parameters of the Reggeon field theory depend on the embedding environment and are chosen as constants; thus the environment is considered uniform and its fluctuations are ignored there.

The DP hypothesis and the associated DP universality class are believed to be very general and robust. Nonetheless, it is reasonable to expect that environmental fluctuations should affect the universal scaling properties of the AAPT in the DP universality class. For instance, the critical scaling behaviour of a density  $\phi$  undergoing an AAPT in the presence of fluctuating environments has been shown in [9]; see also [10, 11] for related studies. In [9] different models were used to describe the fluctuating environments namely (i) the randomly stirred fluid modeled by the Navier-Stokes equation and (ii) fluctuating surface modelled by the Kardar-Parisi-Zhang equation or (iii) the Edward-Wilkinson equation. In all these cases the dynamic exponent of the environment was found to be either same as the DP dynamic exponent (strong dynamic scaling) or different from that of the dynamic exponent of the percolating field (weak dynamic scaling) resulting in non DP behaviour. Not surprisingly, critical exponents belonging to new universality classes were found. From a technical perspective, in all these examples (a) the environment is modeled by a long-ranged noise driven conserved hydrodynamic variable, e.g., a velocity field or a fluctuating surface (equivalently a Burgers velocity field), and (b) in all the cases the feedback of the density field undergoing AAPT on the environment is ignored, i.e., the dynamics of the environment is assumed to be *autonomous*. Both of these features are certainly special cases, since the origin of environmental fluctuations may be very different from a fluctuating conserved variable (e.g., a Navier-Stokes velocity field). For instance, the environment may contain *broken symmetry fields*, e.g., elastic deformations of crystals or liquid crystals, fluctuations in membranes, deformations in an ordered suspension of orientable particles etc. Secondly, the dynamics of the environment, in general, is *not* expected to be autonomous; instead it should be affected by the density  $\phi$  that undergoes an AAPT. Since  $\phi$  is expected to have a long-ranged correlation near the AAPT transition, it will *effectively* act as an *additional stochastic noise source* with long-ranged correlation, which may alter the scaling properties of the environmental dynamics. These issues are likely to be important in some recent experiments on living cells [12], discussing two possible symmetry-determined orientationally ordered states: (a) the *active vectorial* or *polar* order, where the (elongated) cells are oriented along

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a mean direction  $\hat{\mathbf{p}}$  with  $\hat{\mathbf{p}}$  and  $-\hat{\mathbf{p}}$  being *inequivalent*, and (b) *active apolar* or *nematic* order where  $\hat{\mathbf{p}}$  and  $-\hat{\mathbf{p}}$  are equivalent. At the continuum mesoscopic level, these systems are described typically by coupled dynamical equations of the particle density and local orientational order parameter (and also a hydrodynamic velocity if the system is momentum conserving); see, e.g., see Ref. [13] for recent reviews and detailed discussions concerning these systems. A similar example is the orientational order of the magnetotactic bacteria along the earth's magnetic field lines [14]. In such systems, if the experimental time scales are much larger than the natural birth (reproduction/cell division) and death time-scales, then such nonconservation processes are likely to affect the emerging macroscopic properties.

Motivated by the theoretical issues of the effects of feedback to the fluctuating environment on the universal scaling properties of AAPT, in this paper we propose and study a simple reduced model for AAPT in the presence of an environment modeled by a broken symmetry field described by a vector field  $\mathbf{v}$ , whose dynamics in turn is affected by  $\phi$  (feedback). Thus, this study is substantially different from and complementary to Ref. [9] in having an environment dynamics that is no longer autonomous due to the feedback, a situation not considered in Ref. [9]. Apart from the consideration of the feedback of  $\phi$  on the dynamics of the environment, we point out a crucial technical difference that the environmental dynamics has a *short-ranged* Gaussian noise (see below), unlike in Ref. [9], where the corresponding noises are all considered to be spatially long-ranged. Our principal result here is that the scaling behaviour of the system near the extinction transition of the AAPT is, in general, affected by the feedback on the environment and hence non-DP like. In general, depending upon the location of the system in the phase space, one may encounter *strong dynamic scaling* (when both  $\phi$  and  $\mathbf{v}$  have the same dynamic exponents relating spatial and temporal scalings) or *weak dynamic scaling*, when the two dynamic exponents are different. Our results here should help us understand the general effects of mutual dynamical interaction between a density field and the embedding environment for more realistic but complicated situations. The rest of the paper is organised as follows: In Sec. II, we set up our model following a brief review the DP universality class. Then we do a detailed dynamic renormalisation group (DRG) analysis of our model to obtain the scaling exponents at the AAPT in Sec. III. Finally, in Sec. IV we conclude and summarise our results.

## II. DYNAMICAL MODEL

In order to address the issues as mentioned above systematically we construct a simple model in which a density field  $\phi$  undergoing AAPT is coupled to a fluctuating broken symmetry field, represented by a vector field  $\mathbf{v}$ , which acts as the environment. A feedback from the density  $\phi$  to the dynamics of  $\mathbf{v}$  is the distinguishing feature of the present model. Before we discuss it in details, in order to set up the background, we briefly review the problem of extinction transition of a single species in a uniform environment and the scaling exponents at the corresponding AAPT as described by the DP universality class or Gribov process.

### A. Directed Percolation model

Let us consider a population dynamics with a population growth rate depending linearly on the local species density and a death rate controlled by the square of the local density (qualitatively representing death due to overcrowding) undergoing a non-equilibrium active to absorbing state (i.e., species extinction) phase transition whose long distance large time properties are well-described by the DP universality class. In terms of a local particle density  $\phi(\mathbf{x}, t)$ , the Langevin equation that describes such a population dynamics is given by [see, e.g., Ref. [1]]

$$\frac{\partial \phi}{\partial t} = D \nabla^2 \phi + \lambda_g \phi - \lambda_d \phi^2 + \sqrt{\phi} \zeta, \quad (1)$$

where diffusive modes of the density is included with  $D$  as the diffusion coefficient,  $\lambda_g$  is the growth rate and  $\lambda_d$  the decay rate. Stochastic function  $\zeta(\mathbf{x}, t)$  is a zero-mean, Gaussian distributed white noise with a variance

$$\langle \zeta(\mathbf{x}, t) \zeta(0, 0) \rangle = 2D_2 \delta(\mathbf{x}) \delta(t). \quad (2)$$

The in-principle existence of an absorbing state ( $\phi = 0$ ) in the system is ensured by the multiplicative nature of the effective noise. We may extract the characteristic length  $\xi \sim \sqrt{D/|\lambda_g|}$  and diffusive time scale  $t_c \sim \xi^2/D \sim 1/|\lambda_g|$  on dimensional ground, from Eq. (1), both of which diverge upon approaching the critical point at  $\lambda_g = 0$ . We then define the critical exponents in the usual way [1]

$$\langle \phi(\mathbf{x}, t \rightarrow \infty) \rangle \sim \lambda_g^\beta, \quad \langle \phi(\mathbf{x}, t) \rangle \sim t^{-\alpha} (\lambda_g = 0), \quad \xi \sim \lambda_g^{-\nu}, \quad t_c \sim \xi_\phi^z / D \sim \lambda_g^{-z_\phi \nu}, \quad (3)$$

yielding the mean-field values for the scaling exponents

$$\beta = 1, \alpha = 1, \nu = 1/2, \text{ and, } z_\phi = 2. \quad (4)$$

In addition, the anomalous dimension  $\eta$ , which characterises the spatial scaling of the two-point correlation function, is zero [1]. Whether or not fluctuations change the scaling behaviour of the DP problem from their mean-field values, characterised by (4) is an important question here. The DP problem, as modeled by Janssen-de Dominicis action functional corresponding to the Langevin Eq. (1), is invariant under the rapidity symmetry given by  $\hat{\phi}(\mathbf{x}, t) \leftrightarrow \phi(\mathbf{x}, -t)$  [1], where  $\hat{\phi}$  is the dynamic conjugate variable [1]; see below also. This invariance formally defines the DP universality class. All models belonging to the DP universality class are invariant under the rapidity symmetry asymptotically. In order to account for the fluctuation effects, which are expected to affect the mean-field exponent values (4), DRG calculations have been performed over an equivalent path integral description of the Langevin Eq. (1) [1]. By using one-loop renormalised theory with a systematic  $\epsilon$ -expansion,  $\epsilon = d_c - d$ , where the upper critical dimension  $d_c = 4$  for this model, one obtains [1],

$$z = 2 - \epsilon/12, \eta = \epsilon/12 \text{ and } \frac{1}{\nu} = 2 + \epsilon/4. \quad (5)$$

The set of exponents (5) formally constitute and characterise the DP universality class. Recent studies suggest that the DP universality class is fairly robust, a feature formally known as the directed percolation (DP) hypothesis [2]. Only when one or more conditions of the DP hypothesis are violated, one finds new universal properties. For instance, the presence of long range interactions are known to modify the scaling behaviour: Ref. [15] examines the competition between short and long ranged interactions, and identified four different possible phases. Subsequently, Refs. [9–11] have shown how fluctuating environments driven by spatially long-ranged noises (but with autonomous dynamics) may modify the scaling behaviour of the DP universality. In the present work, we extend and complement the existing results by considering a model study (without any long-ranged noise) that considers the effects of feedback of the species density on the environment dynamics. It is expected that such additional couplings between the species density and the environment may alter the universal behaviour at the AAPT. Our perturbative results below confirm this.

## B. Extinction transition in the presence of a broken symmetric field

Having set up the background of our work here, in this subsection we set up the equations of motion for our model of the the density field  $\phi(\mathbf{x}, t)$  undergoing AAPT coupled with the broken symmetry field  $\mathbf{v}(\mathbf{x}, t)$  in the hydrodynamic limit, retaining minimal but relevant coupling terms connecting the dynamics of  $\phi$  and  $\mathbf{v}$ . A broken symmetry field, also known as a *Goldstone variable* in the literature is a deformation of an ordered state that originates in a system due to the breakdown of a continuous symmetry. Well-known examples of broken symmetry states include crystals (broken translational invariance), nematic liquid crystals (broken rotational invariance), Heisenberg ferromagnetic systems (broken rotational invariance in the order parameter space) [16]. A broken symmetry mode necessarily has a life-time of a fluctuation that diverges in the zero wavevector limit, reflecting the simple fact that cost of configuration energy associated with the creation of a broken symmetry mode with a given wavelength vanishes as the wavelength of the fluctuation diverges. There are, however, no conservation law associated with a broken symmetry variable. A broken symmetry variable may have a variety of symmetry, depending upon the actual physical system concerned. For instance, the local displacement fields [16, 17], the broken symmetry variables in a crystal are invariant under shifts by constant amounts, whereas, the Frank director field, which are the relevant broken symmetry variables in a nematic liquid crystal, are invariant under a combined rotation of the coordinate system and the director fields. From a general theoretical point of view, it is interesting to study the universal behaviour of AAPT in contact with a broken symmetry mode. Apart from this, studies on AAPT in contact with broken symmetries are potentially relevant in exploring the universal properties of the extinction transitions in a bacteria colony populated by bacteria in their orientated states, e.g., polar or nematic. The time-evolution of the polar or nematic order parameter should be generically coupled to the density undergoing AAPT, and hence may affect the scaling at the AAPT. Since the order parameter field is a broken symmetry field, its correlation function is scale invariant, displaying universal scaling. Whether the scale invariant density field at the AAPT modifies the scaling of the order parameter fields through the mutual dynamical couplings is an associated relevant question. While we are motivated by these examples, in the present work, we do not intend to model a specific case of broken symmetry variable as the environment; rather, it may be considered as a toy model for AAPT in contact with a broken symmetry variable with a simple structure. To this effect, we enforce a simple invariance on the broken symmetry variable  $\mathbf{v}$  in the model by demanding invariance under  $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{v}_0$  [18]. Thus any coupling between  $\phi$  and  $\mathbf{v}$  should involve  $\nabla \cdot \mathbf{v}$ . Such considerations allow us to write down the dynamical equation for  $\phi$ : This is essentially same as Eq. (1), supplemented by a symmetry-allowed

coupling term involving  $\nabla \cdot \mathbf{v}$  and  $\phi$ . The resulting equation of motion for  $\phi$  up to the lowest order in spatial gradients takes the form

$$\frac{\partial \phi}{\partial t} = \lambda_g \phi - \lambda_d \phi^2 + D \nabla^2 \phi + \lambda_1 \phi \nabla \cdot \mathbf{v} + \sqrt{\phi} \xi, \quad (6)$$

where  $\lambda_1$  is the coefficient describing the most dominant lowest order coupling that couples a vector field  $\mathbf{v}$  with a scalar field  $\phi$ ,  $\lambda_g$  and  $\lambda_d$  are the growth and decay coefficients of the density  $\phi$  and  $\xi$  is the gaussian distributed white noise with a variance as given by Eq. (2).

To complete the dynamical description of our model, we now need a corresponding equation for  $\mathbf{v}$ . We use a simple relaxational dynamics for  $\mathbf{v}$ . To obtain the appropriate dynamical equation, we start with a free energy functional: We assume that the energy associated with the configurations of  $\mathbf{v}$  are given by

$$\mathcal{F} = \frac{1}{2} \int d^d x [\lambda (\nabla_i v_j)^2 + 2\chi (\nabla \cdot \mathbf{v}) \phi], \quad (7)$$

where  $\lambda > 0$  is the stiffness modulus (akin to the elastic moduli for a crystal or the Frank elastic constants for nematic liquid crystals) and  $\chi$  is the coupling constant for the bilinear coupling between  $\mathbf{v}$  and  $\phi$ . Assuming a non-conserved relaxational dynamics (model A in the language of Ref. [19]) the stochastically driven Langevin equation for  $\mathbf{v}$  becomes  $\frac{\partial v_i}{\partial t} = -\hat{\Gamma} \frac{\delta \mathcal{F}}{\delta v_i} + f_i$ , where  $f_i$  is a zero-mean Gaussian noise,  $\hat{\Gamma}$  is a kinetic coefficient (set to unity below). With the choice of  $\mathcal{F}$  as above, we find

$$\frac{\partial v_i}{\partial t} = \lambda \nabla^2 v_i + \chi \nabla_i \phi + f_i. \quad (8)$$

For systems in equilibrium, the variance of  $f_i$  would have been related to it through the Fluctuation-Dissipation-Theorem (FDT) [16]. However, the present system being out-of-equilibrium, where there is no FDT, the variance of  $\mathbf{f}$  is unrelated to  $\hat{\Gamma}$ . We choose

$$\langle f_i(\mathbf{x}, t) f_j(0, 0) \rangle = 2D_0 \delta^d(x) \delta(t) \delta_{ij}. \quad (9)$$

From symmetry point of view, clearly, our model Eqs. (6) and (8) are not invariant under  $\mathbf{v} \rightarrow -\mathbf{v}$ . Thus, this is reminiscent of polar (or vectorial) symmetry of Ref. [20].

The vector field  $v_i$  being a broken symmetry field has a dynamics that is generically scale invariant, characterised by a set of scaling exponents. They are defined via the correlation function

$$\langle v_i(\mathbf{x}, t) v_j(0, 0) \rangle = |\mathbf{x}|^{2-d-\eta_v} \psi_{ij}^v(|\mathbf{x}|^{z_v}/t), \quad (10)$$

where  $\eta_v$  and  $z_v$  are the anomalous dimension and dynamic exponent, respectively of  $\mathbf{v}$ , and  $\psi_{ij}^v$  is a dimensionless scaling function of its argument. Ignoring the coupling with  $\phi$ , exponents  $\eta_v = 0$  and  $z_v = 2$  are known exactly. Whether the coupling with  $\phi$  alters these exponents is a question that we address here within a one loop perturbative calculation.

Redefining coefficient  $\lambda_g = D\tau$  and  $\lambda_d = \frac{Dg_2}{2}$  for calculational convenience, Eq. (6) may be written as

$$\frac{\partial \phi}{\partial t} = D(\tau + \nabla^2) \phi - \frac{Dg_2}{2} \phi^2 + \lambda_1 (\nabla \cdot \mathbf{v}) \phi + \sqrt{\phi} \xi, \quad (11)$$

which redefines the critical point as renormalised  $\tau = 0$ . In the mean field picture (dropping all nonlinearities) at  $\tau = 0$ , density  $\phi$  undergoes an AAPT displaying the mean-field DP universal behaviour with critical exponents given by Eq. (4) above. Whether or not the nonlinear coupling terms  $\lambda_1$  and  $Dg_2$ , together with the (linear) feedback term with coefficient  $\chi \hat{\Gamma}$  are able to alter the mean-field universal behaviour can only be answered by solving the full coupled equations (11) and (8). Their overall nonlinear nature rules out the possibility of any exact solution. A well-established framework for addressing this issue systematically is the standard implementation of DRG procedure, based on a one-loop perturbative expansion in the coupling constants  $\lambda_1$  and  $Dg_2$  about the linear theory. The resulting perturbative corrections of the different (bare) model parameters may then be used to construct the renormalised correlation functions.

We begin with the Janssen-De Dominicis generating functional [21] corresponding to the Langevin Eqs. (8) and (11) and the noise variances (2) and (9), which allows us to describe the dynamics as a path integral over the relevant dynamical fields in the system. For the convenience of calculations that follow, we redefine  $i\hat{v}_i \rightarrow \hat{v}_i$ ,  $i\hat{\phi} \rightarrow \beta_1 \hat{\phi}$  and  $\phi \rightarrow \beta_2 \phi$ ,  $\beta_1^2 D_2 \beta_2 = \frac{Dg_1}{2}$ ,  $\beta_1 \beta_2 = 1$ ,  $\lambda_g = \tau D$  and  $\lambda_d \beta_2 = \frac{Dg_2}{2}$  [22]. Writing the generating functional as

$$\langle \mathcal{Z} \rangle_f = \int D\phi D\hat{\phi} Dv D\hat{v} \exp[-\mathcal{S}], \quad (12)$$

where  $\mathcal{S}$  is the action functional of the system. The expression for  $\mathcal{S}$  can be written as

$$\begin{aligned} \mathcal{S} = & -\frac{Dg_1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{d\Omega}{2\pi} \hat{\phi}_{-\mathbf{k},-\omega} \hat{\phi}_{\mathbf{q},\Omega} \phi_{\mathbf{k}-\mathbf{q},\omega-\Omega} + \frac{Dg_2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{d\Omega}{2\pi} \hat{\phi}_{-\mathbf{k},-\omega} \phi_{\mathbf{q},\Omega} \phi_{\mathbf{k}-\mathbf{q},\omega-\Omega} \\ & + \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \hat{\phi}_{-\mathbf{k},-\omega} \left\{ i\omega \phi_{\mathbf{k},\omega} + D(-\tau + k^2) \phi_{\mathbf{k},\omega} - i\lambda_1 \int \frac{d^d q}{(2\pi)^d} \frac{d\Omega}{2\pi} \hat{\phi}_{-\mathbf{k},-\omega} q_l v_l(\mathbf{q},\Omega) \phi_{\mathbf{k}-\mathbf{q},\omega-\Omega} \right\} \\ & - \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} D_0 \hat{v}_i(-\mathbf{k},-\omega) \hat{v}_i(\mathbf{k},\omega) + \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \hat{v}_i(-\mathbf{k},-\omega) \{ (i\omega + \lambda k^2) v_i(\mathbf{k},\omega) - i\chi k_i \phi_{\mathbf{k},\omega} \}. \end{aligned} \quad (13)$$

Here  $\mathbf{k}, \mathbf{q}$  represent momenta and  $\omega, \Omega$  represent frequencies in the Fourier space. The first two terms in Eq. (13) have different coefficients  $\frac{Dg_1}{2}$  and  $\frac{Dg_2}{2}$  which shows the breakdown of invariance under rapidity symmetry [1] as a result of the couplings  $\lambda_1$  and  $\chi$ . Notice that by rescaling time we may absorb the coefficient  $\lambda$ . This explains the lack of renormalization for  $\lambda$  (see below for details).

Before we embark upon the detailed calculation, let us note the following: First of all, the action functional (13) is no longer invariant under the rapidity symmetry; the coupling with the broken symmetry field  $\mathbf{v}$  explicitly breaks it. Given our wisdom from equilibrium critical phenomena and equilibrium critical dynamics, new universal behaviour is expected, provided the dynamical couplings between  $v_i$  and  $\phi$  are relevant. As a result, scaling exponents should have values different from their values in the DP universality class given by (5). When the coupling  $\chi$  [the feedback term in Eq. (8)] is zero, the dynamics of  $\mathbf{v}$  becomes autonomous, i.e., independent of  $\phi$ . Evidently, in this case,  $z_v = 2$ , where as  $z_\phi$  may or may not be 2. Thus, one may encounter both *weak* and *strong* dynamic scaling. On the other hand, when (renormalised)  $\chi \neq 0$ , the dynamics of  $\mathbf{v}$  is no longer autonomous; it gets affected by the dynamics of  $\phi$ , such that  $z_v$  may be different from 2. Whether or not  $z_\phi$  is same as  $z_v$  can be ascertained only after a detailed calculation that we present below.

To start with we note that the roles of the (*bare or unrenormalised*) coupling constants in an ordinary perturbative expansion of the present model are played by  $u = g_1 g_2$  and  $w = \frac{\lambda^2}{D^3}$ . Our model has upper critical dimension  $d_c = 4$ , such that both the coupling constants  $u$  and  $w$  become dimensionless at  $d = 4$ , and the mean-field exponents (4) are to provide quantitatively correct description of scaling for  $d \geq 4$ . We set up a renormalised perturbative expansion in  $\epsilon = 4 - d$  up to the one-loop order. To ensure ultra-violet (UV) renormalisation of the present model, we render finite all the non-vanishing two-, three-point vertex functions by introducing multiplicative renormalisation constants. This procedure is standard and well-documented in the literature, see, e.g., Ref. [23]. Here, the vertex functions of different orders are formally defined by appropriate functional derivatives of the vertex generating functional  $\Gamma[\phi, \hat{\phi}, v_i, \hat{v}_i]$  which is the Legendre transformation of  $\log \mathcal{Z}$  [23]. The bare values of the different vertex functions can be easily read off the action functional (13) and are given by (after separating out the various  $\delta$ -functions associated with spatial and temporal translation invariance)

$$\frac{\delta^2 \Gamma}{\delta \phi(\mathbf{k}, \omega) \delta \hat{\phi}(-\mathbf{k}, -\omega)} = \Gamma_{\phi \hat{\phi}} = i\omega + D(-\tau + k^2), \quad (14)$$

$$\frac{\delta^2 \Gamma}{\delta v_i(\mathbf{k}, \omega) \delta \hat{v}_j(-\mathbf{k}, -\omega)} = \Gamma_{v_i \hat{v}_j} = (i\omega + \lambda k^2) \delta_{ij}, \quad (15)$$

$$\frac{\delta^2 \Gamma}{\delta \hat{v}_i(\mathbf{k}, \omega) \delta \hat{v}_j(-\mathbf{k}, -\omega)} = \Gamma_{\hat{v}_i \hat{v}_j} = 2D_0 \delta_{ij}, \quad (16)$$

$$\frac{\delta^2 \Gamma}{\delta \hat{v}_i(-\mathbf{k}, -\omega) \delta \phi(\mathbf{k}, \omega)} = \Gamma_{\hat{v}_i \phi} = -i\chi k_i, \quad (17)$$

$$\frac{\delta^3 \Gamma}{\delta \hat{\phi}(\mathbf{q}_1, \omega_1) \delta \hat{\phi}(\mathbf{q}_2, \omega_2) \delta \phi(-\mathbf{q}_1 - \mathbf{q}_2, -\omega_1 - \omega_2)} = \Gamma_{\hat{\phi} \hat{\phi} \phi} = -\frac{Dg_1}{2}, \quad (18)$$

$$\frac{\delta^3 \Gamma}{\delta \hat{\phi}(\mathbf{q}_1, \omega_1) \delta \phi(\mathbf{q}_2, \omega_2) \delta \phi(-\mathbf{q}_1 - \mathbf{q}_2, -\omega_1 - \omega_2)} = \Gamma_{\hat{\phi} \phi \phi} = \frac{Dg_2}{2}, \quad (19)$$

$$\frac{\delta^3 \Gamma}{\delta v_i(\mathbf{k}, \omega) \delta \hat{\phi}(\mathbf{q}, \Omega) \delta \phi(-\mathbf{k} - \mathbf{q}, -\omega - \Omega)} = \Gamma_{v_i \hat{\phi} \phi} = -i\lambda_1 k_i. \quad (20)$$

### III. RENORMALISATION GROUP CALCULATIONS AND THE SCALING EXPONENTS

In order to renormalise the vertex functions by carrying out the one loop integrals we choose  $\tau = \mu^2$  as our appropriate normalization point, where  $\mu$  is an intrinsic momentum scale of the renormalised theory. This will allow

us to find the scale dependence of the renormalised correlation or vertex functions on  $\mu$  by using the multiplicative  $Z$ -factors for the fields and the parameters in the model. These  $Z$ -factors are useful in absorbing all the ultraviolet divergences arising from the one loop diagrammatic corrections thus giving us an effective finite theory. Formally, the  $Z$ -factors present in this model are defined as

$$\begin{aligned}\phi &= Z_\phi \phi_R, \quad v = Z_v v_R, \quad \hat{v} = Z_{\hat{v}} \hat{v}_R, \quad \hat{\phi} = Z_{\hat{\phi}} \hat{\phi}_R, \quad D = Z_D D_R, \quad \lambda_1 = Z_{\lambda_1} \lambda_{1R}, \quad g_1 = Z_{g_1} g_{1R}, \quad g_2 = Z_{g_2} g_{2R}, \quad \tau = Z_\tau \tau_R, \\ \lambda &= Z_\lambda \lambda_R, \quad \chi = Z_\chi \chi_R,\end{aligned}\tag{21}$$

where a subscript  $R$  refers to a renormalised quantity. The different  $Z$ -factors may be enumerated from the following conditions on the renormalised vertex functions:

$$\frac{\partial \Gamma_{\hat{\phi}\phi}}{\partial \omega} \Big|_{(\mathbf{k}=0, \omega=0)} = i,\tag{22}$$

$$\frac{\partial \Gamma_{\hat{\phi}\phi}}{\partial k^2} \Big|_{(\mathbf{k}=0, \omega=0)} = D_R,\tag{23}$$

$$\Gamma_{\hat{\phi}\phi}(\mathbf{k} = 0, \omega = 0) = D_R \tau_R,\tag{24}$$

$$\frac{\partial \Gamma_{\hat{v}_i v_j}}{\partial \omega} \Big|_{(\mathbf{k}=0, \omega=0)} = i \delta_{ij},\tag{25}$$

$$\frac{\partial \Gamma_{\hat{v}_i v_j}}{\partial k^2} \Big|_{(\mathbf{k}=0, \omega=0)} = \lambda \delta_{ij},\tag{26}$$

$$\Gamma_{\hat{\phi}\hat{\phi}\phi}(\mathbf{k} = 0, \mathbf{q} = 0, \omega = 0, \Omega = 0) = -\frac{D_R g_{1R}}{2},\tag{27}$$

$$\Gamma_{\hat{\phi}\hat{\phi}\phi}(\mathbf{k} = 0, \mathbf{q} = 0, \omega = 0, \Omega = 0) = \frac{D_R g_{2R}}{2},\tag{28}$$

$$\frac{\partial}{\partial k_i} \Gamma_{v_i \hat{\phi}\phi} \Big|_{(\mathbf{k}=0, \mathbf{q}=0, \omega=0, \Omega=0)} = -i \lambda_{1R},\tag{29}$$

$$\Gamma_{\hat{v}_i \hat{v}_j}(\mathbf{k} = 0, \omega = 0) = -2D_0 \delta_{ij}.\tag{30}$$

There are 11  $Z$ -factors defined in Eq. (21) above, as compared to the 9 renormalisation conditions on the renormalised vertex functions, as given in Eq. (30). Thus, two of the  $Z$ -factors defined above are redundant. Without any loss of generality, we set  $Z_\phi = Z_{\hat{\phi}}$  and  $Z_v = Z_{\hat{v}}$ . Explicit forms for the  $Z$ -factor are given by

$$\begin{aligned}Z_\phi &= Z_{\hat{\phi}} = 1 + \frac{g_1 g_2 \mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} - \frac{D_0 \lambda_1^2 \mu^{-\epsilon}}{\lambda(\lambda + D)^2 \epsilon} \frac{1}{16\pi^2} - \frac{\lambda_1 \chi g_1 (3D + \lambda) \mu^{-\epsilon}}{4D(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2}, \\ Z_D &= 1 - \frac{g_1 g_2 \mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1 \chi g_1 (7D^2 + 4\lambda D + \lambda^2) \mu^{-\epsilon}}{4D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2} + \frac{2DD_0 \lambda_1^2 \mu^{-\epsilon}}{\lambda(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2}, \\ Z_\tau &= 1 + \frac{3g_1 g_2 \mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} - \frac{2DD_0 \lambda_1^2 \mu^{-\epsilon}}{\lambda(D + \lambda)^3} \frac{1}{16\pi^2} - \frac{\lambda_1 \chi g_1 (5D + 3\lambda) \mu^{-\epsilon}}{2D(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} - \frac{\lambda_1 g_1 \chi (7D^2 + 4D\lambda + \lambda^2) \mu^{-\epsilon}}{4D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2}, \\ Z_{g_2} &= 1 + \frac{3g_1 g_2 \mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} - \frac{D_0 \lambda_1^2 (4\lambda + 5D) \mu^{-\epsilon}}{\lambda D(\lambda + D)^2 \epsilon} \frac{1}{16\pi^2} - \frac{2DD_0 \lambda_1^2 \mu^{-\epsilon}}{\lambda(\lambda + D)^3 \epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1^2 g_1 \chi^2 (2D + \lambda) \mu^{-\epsilon}}{\lambda D^2 g_2 (\lambda + D)^2 \epsilon} \frac{1}{16\pi^2} \\ &\quad + \frac{\lambda_1 g_1 \chi (3D + \lambda) \mu^{-\epsilon}}{D(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} - \frac{4\lambda_1^3 \chi D_0 \mu^{-\epsilon}}{\lambda^2 D^2 g_2 (D + \lambda) \epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1 \chi g_1 (5D^2 + 12\lambda D + 5\lambda^2) \mu^{-\epsilon}}{2D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2}, \\ Z_{g_1} &= 1 + \frac{3g_1 g_2 \mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} - \frac{D_0 \lambda_1^2 (5D + 4\lambda) \mu^{-\epsilon}}{D\lambda(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} - \frac{2DD_0 \lambda_1^2 \mu^{-\epsilon}}{\lambda(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1 \chi g_1 (9D^2 + 16D\lambda + 5\lambda^2) \mu^{-\epsilon}}{2D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2}.\end{aligned}\tag{31}$$

There are no one-loop corrections to  $\chi$  and  $\lambda$  [24]. To find out the  $Z$ -factor corresponding to  $\lambda_1$ , we first set  $\chi = \frac{D^2 g_2 \alpha}{\lambda_1}$  without any loss of generality, where  $\alpha$  is a dimensionless number. From the relation  $\Gamma_{\hat{v}\phi}^R = iZ_{\hat{v}} Z_\phi k_i \chi = i k_i \frac{D_R^2 g_{2R} \alpha_R}{\lambda_{1R}} = i \chi k_i Z_D^{-2} Z_{g_2}^{-1} Z_{\lambda_1} Z_\alpha^{-1}$ , we find  $Z_{\hat{v}} = Z_D^{-2} Z_{g_2}^{-1} Z_{\lambda_1} Z_\alpha^{-1} Z_{\hat{\phi}}^{-1}$ . Now from  $\Gamma_{\hat{v}_i v_i}^R(\mathbf{k} = 0) = i\omega Z_v Z_{\hat{v}} = i\omega$  it can be easily seen that  $Z_v Z_{\hat{v}} = 1$  or  $Z_v = Z_{\hat{v}}^{-1}$ . This lets us write  $Z_{\hat{v}} = Z_v^{-1} = Z_D^{-2} Z_{g_2}^{-1} Z_{\lambda_1} Z_\alpha Z_{\hat{\phi}}^{-1}$ . Next, using this relation in the expression  $\Gamma_{v_i \hat{\phi}\phi}^R = -i k_i \lambda_1 Z_{\lambda_1}^{-1}$ , we get

$$\begin{aligned}Z_\alpha &= 1 - \frac{3g_1 g_2 \mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} - \frac{\lambda_1 g_1 \chi (6D^2 + 8\lambda D + 3\lambda^2) \mu^{-\epsilon}}{D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2} + \frac{2\lambda_1^2 D_0 (2D^2 + 2\lambda^2 + 5D\lambda)}{\lambda D(D + \lambda)^3 \epsilon} \frac{1}{16\pi^2} \\ &\quad + \frac{\lambda_1 g_1 \chi (5D + 3\lambda) \mu^{-\epsilon}}{4D(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} - \frac{\lambda_1^2 g_1 \chi^2 (2D + \lambda) \mu^{-\epsilon}}{D^2 \lambda g_2 (D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} + \frac{4\lambda_1^3 D_0 \chi \mu^{-\epsilon}}{\lambda^2 D^2 g_2 (D + \lambda) \epsilon} \frac{1}{16\pi^2}.\end{aligned}\tag{32}$$

Equation (32), together with  $Z_v = Z_{\hat{v}} = 1$  yield

$$Z_{\lambda_1} = 1 + \frac{g_1 g_2 \mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1 g_1 \chi (3D + \lambda) \mu^{-\epsilon}}{D(D + \lambda)^2 \epsilon} \frac{1}{16\pi^2} + \frac{\lambda_1 g_1 \chi \mu^{-\epsilon}}{2D(D + \lambda) \epsilon} \frac{1}{16\pi^2}. \quad (33)$$

Further, define  $u = g_1 g_2$  and  $w = \frac{\lambda_1^2 D_0}{D^3}$ , the  $Z$  factors for them being  $Z_u = Z_{g_1} Z_{g_2}$  and  $Z_w = \frac{Z_{\lambda_1}^2}{Z_D^3}$ ; with  $\theta = \frac{\lambda}{D}$ ,  $Z_\theta = Z_D^{-1}$  as  $\lambda$  does not renormalise in the model. Formally,  $Z$ -factors (31,32) and (33) may be used to define the  $\beta$ -functions for the renormalised coupling constants  $u_R, w_R, \alpha_R, \theta_R$ . We obtain (after absorbing  $1/16\pi^2$  in the definitions of the renormalised coupling constants)

$$\beta_u = u_R \left[ \frac{3u_R}{2} - \frac{2w_R(5 + 4\theta_R)}{\theta_R(1 + \theta_R)^2} - \frac{4w_R}{\theta_R(1 + \theta_R)^3} + \frac{\alpha_R^2 u_R(2 + \theta_R)}{\theta_R(1 + \theta_R)^2} - \frac{4w_R \alpha_R}{\theta_R^2(1 + \theta_R)} + \frac{\alpha_R u_R(3 + \theta_R)}{(1 + \theta_R)^2} + \frac{\alpha_R u_R}{(1 + \theta_R)^3} (7 + 14\theta_R + 5\theta_R^2) - \epsilon \right], \quad (34)$$

$$\beta_w = w_R \left[ \frac{7u_R}{8} + \frac{\alpha_R u_R(7 + 3\theta_R)}{(1 + \theta_R)^2} - \frac{3\alpha_R u_R}{4(1 + \theta_R)^3} (7 + 4\theta_R + \theta_R^2) - \frac{6w_R}{\theta_R(1 + \theta_R)^3} - \epsilon \right], \quad (35)$$

$$\beta_\alpha = \alpha_R \left[ -\frac{3u_R}{8} - \frac{\alpha_R u_R}{(1 + \theta_R)^3} (6 + 8\theta_R + 3\theta_R^2) + \frac{2w_R(2 + 2\theta_R^2 + 5\theta_R)}{\theta_R(1 + \theta_R)^3} + \frac{\alpha_R u_R(5 + 3\theta_R)}{4(1 + \theta_R)^3} - \frac{\alpha_R^2 u_R(2 + \theta_R)}{\theta_R(1 + \theta_R)^2} + \frac{4\alpha_R w_R}{\theta_R^2(1 + \theta_R)} \right], \quad (36)$$

$$\beta_\theta = \theta_R \left[ \frac{u_R}{8} - \frac{\alpha_R u_R(7 + 4\theta_R + \theta_R^2)}{4(1 + \theta_R)^3} - \frac{2w_R}{\theta_R(1 + \theta_R)^3} \right]. \quad (37)$$

The zeros of the  $\beta$ -functions (34-37) above should yield the fixed points (FPs). Physically, there are three possible FP values for  $\theta_R$ :  $\theta_R = 0, \infty$  and  $\theta_R$  finite. The first two should yield  $z_\phi \neq z_v$  (weak dynamic scaling) and the last one  $z_\phi = z_v$  (strong dynamic scaling). In principle, FPs for all the three physical regimes may be obtained from solutions of the respective  $\beta$ -functions. However, due to the complicated nature of Eqs. (34-37), the ensuing algebra is rather involved, precluding full general solutions in closed forms. Notice that the  $Z$ -factors (31, 32, 33) simplify considerably in the limit  $\theta \rightarrow 0$  as shown below, corresponding to  $\theta_R \rightarrow 0$  in the renormalised theory. Instead of obtaining the FPs with arbitrary values of  $\theta_R$ , we obtain the FPs in the limit  $\theta_R \rightarrow 0$  only.

#### A. Analysis in the limit $\theta \rightarrow 0$

By definition  $\theta = \lambda/D$  or in terms of the renormalised quantities  $\theta_R = \lambda_R/D_R$ . Thus, for  $\theta_R \rightarrow 0$ ,  $D_R \gg \lambda_R \Rightarrow z_\phi < z_v$ , i.e., the dynamic exponents of the two interacting field  $\phi$  and  $\mathbf{v}$  are unequal. Thus, this corresponds to *weak dynamic scaling*. Physically meaningful stable solution (corresponding to a stable FP) should then reveal weak dynamic scaling with  $z_\phi < z_v$ . In order to obtain the FPs for  $\theta_R = 0$ , we consider the forms for the  $Z$ -factors (31, 32, 33) in the limit  $\theta \rightarrow 0$  (or, equivalently, the limit  $\theta_R \rightarrow 0$  in the renormalised theory). The  $Z$ -factors naïvely reduce to

$$\begin{aligned} Z_u &= 1 + \frac{3u\mu^{-\epsilon}}{2\epsilon} \frac{1}{16\pi^2} - \frac{14w\mu^{-\epsilon}}{\theta_R\epsilon} \frac{1}{16\pi^2} + \frac{10u\alpha\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} - \frac{4w\alpha\mu^{-\epsilon}}{\theta^2\epsilon} \frac{1}{16\pi^2} + \frac{2u\alpha^2\mu^{-\epsilon}}{\theta\epsilon} \frac{1}{16\pi^2}, \\ Z_w &= 1 + \frac{7u\mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} + \frac{7u\alpha\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} - \frac{6w\mu^{-\epsilon}}{\theta\epsilon} \frac{1}{16\pi^2}, \\ Z_\theta &= 1 + \frac{u\mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} - \frac{7u\alpha\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} - \frac{2w\mu^{-\epsilon}}{\theta\epsilon} \frac{1}{16\pi^2}, \\ Z_\alpha &= 1 - \frac{3u\mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} - \frac{19u\mu^{-\epsilon}\alpha}{4\epsilon} \frac{1}{16\pi^2} + \frac{4w\mu^{-\epsilon}}{\theta\epsilon} \frac{1}{16\pi^2} - \frac{2u\mu^{-\epsilon}\alpha^2}{\theta\epsilon} \frac{1}{16\pi^2} + \frac{4w\mu^{-\epsilon}\alpha}{\theta^2\epsilon} \frac{1}{16\pi^2}, \\ Z_{\lambda_1} &= 1 + \frac{u\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} + \frac{3u\alpha\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} + \frac{u\alpha\mu^{-\epsilon}}{2\epsilon} \frac{1}{16\pi^2}. \end{aligned} \quad (38)$$

The corresponding  $\beta$ -functions for the renormalised coupling constants  $u_R, w_R, \theta_R$  and  $\alpha_R$  reduce to

$$\beta_u = u_R[-\epsilon + \frac{3}{2}u_R - \frac{14w_R}{\theta_R} + 10u_R\alpha_R - 4\frac{w_R\alpha_R}{\theta_R^2} + 2\frac{u_R\alpha_R^2}{\theta_R}], \quad (39)$$

$$\beta_w = w_R[-\epsilon + \frac{7u_R}{8} + \frac{7}{4}u_R\alpha_R - \frac{6w_R}{\theta_R}], \quad (40)$$

$$\beta_\alpha = \alpha_R[-\frac{3u_R}{8} - \frac{19u_R\alpha_R}{4} + \frac{4w_R}{\theta_R} - \frac{2u_R\alpha_R^2}{\theta_R} + \frac{4w_R\alpha_R}{\theta_R^2}], \quad (41)$$

$$\beta_\theta = \theta_R[\frac{u_R}{8} - \frac{7}{4}u_R\alpha_R - \frac{2w_R}{\theta_R}]. \quad (42)$$

Since we consider the FPs in the  $\theta_R \rightarrow 0$  limit, these FPs are given by the zeros of the  $\beta$ -functions (39-41). The non-trivial FPs, for which  $u_R, w_R, \alpha_R \neq 0$  are then given by the equations

$$\frac{3}{2}u_R - \frac{14w_R}{\theta_R} + 10u_R\alpha_R - 4\frac{w_R\alpha_R}{\theta_R^2} + 2\frac{u_R\alpha_R^2}{\theta_R} = \epsilon, \quad (43)$$

$$\frac{7u_R}{8} + \frac{7}{4}u_R\alpha_R - \frac{6w_R}{\theta_R} = \epsilon, \quad (44)$$

$$-\frac{3u_R}{8} - \frac{19u_R\alpha_R}{4} + \frac{4w_R}{\theta_R} - \frac{2u_R\alpha_R^2}{\theta_R} + \frac{4w_R\alpha_R}{\theta_R^2} = 0. \quad (45)$$

Now,  $u_R, w_R \sim O(\epsilon)$  and  $\alpha_R$  is a finite number (or zero) at the FPs with  $\theta_R \rightarrow 0$ . For physically meaningful FPs with finite values of  $u_R, w_R, \alpha_R$  in the limit  $\theta_R \rightarrow 0$ , the  $\beta$ -functions (39-41) should stay finite, or, equivalently, no terms in the equations (43-45) should diverge. Since  $\theta_R$  appears in the denominators of several terms in the  $\beta$ -functions (39-41), naïvely those terms diverge for  $\theta_R \rightarrow 0$ . To prevent this, the respective numerators must scale with  $\theta_R$  appropriately in the limit  $\theta_R \rightarrow 0$ , so that the divergences in the various terms of the  $\beta$ -functions (39-41) [or, in Eqs. (43-45)] cancel out, and all terms in (39-41) or in (43-45) are finite (or zero) in the limit  $\theta_R \rightarrow 0$ .

To proceed further, we assume  $w_R \sim \theta_R^{\gamma_1}$  and  $\alpha_R \sim \theta_R^{\gamma_2}$  in the limit  $\theta_R \rightarrow 0$ , with  $\gamma_1, \gamma_2 > 0$  to be chosen such that the divergences mentioned above are cancelled. Clearly, if  $\gamma_1, \gamma_2$  are too small, some of the terms in Eqs. (39-41) will still diverge for  $\theta_R \rightarrow 0$ . On the other hand, if  $\gamma_1, \gamma_2$  are too large, then *all* terms with  $w_R$  or  $\alpha_R$  will vanish for  $\theta_R \rightarrow 0$ , allowing only the DP FP to survive. Evidently, for finiteness of the  $\beta$ -functions in the limit  $\theta_R \rightarrow 0$  (39-41), we must have  $\gamma_1 \geq 1$ ,  $\gamma_1 + \gamma_2 \geq 2$ ,  $2\gamma_2 \geq 1$ . Non-trivial (non-DP) FPs are obtained, provided one or more of the above inequalities reduce to equalities (i.e., the above conditions hold with the "=" sign, instead of the " $\geq$ " sign). Clearly, all of them cannot hold good simultaneously with the "=" sign. Assuming any two of the above three conditions should hold with "=" sign, i.e., any two of  $w_R/\theta_R, w_R\alpha_R/\theta_R^2, u_R\alpha_R^2/\theta_R$  are not to vanish in the  $\theta_R \rightarrow 0$  there are only two sets of choices for  $\gamma_1, \gamma_2$ , for which non-trivial (non-DP) FPs ensue, while keeping all the  $\beta$ -functions (39-41) above finite. Noting that  $u_R \sim O(\epsilon)$  at the FP, the two choices are as follows:

- Case I:  $\alpha_R \sim \sqrt{\theta_R}, w_R \sim \theta_R^{3/2}\epsilon$ . With this choice,  $\alpha_R^2 u_R/\theta_R \sim O(\epsilon)$  and  $w_R/\theta_R \sim \sqrt{\theta_R}\epsilon \rightarrow 0$  for  $\theta_R \rightarrow 0$ . This corresponds to the bare coupling constants  $w \sim \theta^{3/2}\epsilon, \alpha \sim \sqrt{\theta}$  for  $\theta \rightarrow 0$ .
- Case II:  $w_R \sim \theta_R\epsilon, \alpha_R \sim \theta_R$ . Thus,  $w_R/\theta_R \sim O(\epsilon), \alpha_R w_R/\theta_R^2 \sim O(\epsilon)$  and  $\alpha_R^2 u_R/\theta_R \sim \theta_R\epsilon \rightarrow 0$  for  $\theta_R \rightarrow 0$ . This corresponds to the bare coupling constants  $w \sim \theta\epsilon, \alpha \sim \theta$  when  $\theta \rightarrow 0$ .

We obtain the FPs separately for the two cases above. (Notice that choices for  $\gamma_1, \gamma_2$  such that only one among  $w_R/\theta_R, w_R\alpha_R/\theta_R^2, u_R\alpha_R^2/\theta_R$  is not to vanish in the  $\theta_R \rightarrow 0$  does not lead to any new non-trivial FPs that are already not contained in Case I and Case II above.)

Case I: Evidently, the  $\beta$ -functions (39-41) are linear in  $u_R, u_R\alpha_R^2/\theta_R, w_R\alpha_R/\theta_R^2$ . Equivalently the  $Z$ -factors in (38) are linear in  $u, u\alpha^2/\theta, w\alpha/\theta^2$ . This allows us to identify three effective (bare) coupling constants: (i)  $u$ , (ii)  $s = u\alpha^2/\theta$  (iii)  $b = w\alpha/\theta^2$ . The renormalisation  $Z$ -factors for  $s$  and  $b$  may be calculated in straightforward ways. We obtain

$$\begin{aligned} Z_s &= Z_u Z_\alpha^2 Z_\theta^{-1} = 1 + \frac{5u\mu^{-\epsilon}}{8\epsilon} \frac{1}{16\pi^2} + \frac{4b\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} - \frac{2s\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2}, \\ Z_b &= Z_w Z_\alpha Z_\theta^{-2} = 1 + \frac{u\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} + \frac{4b\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} - \frac{2s\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2}, \\ Z_u &= 1 + \frac{3u\mu^{-\epsilon}}{2\epsilon} \frac{1}{16\pi^2} - \frac{4b\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} + \frac{2s\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2}. \end{aligned} \quad (46)$$



The corresponding  $\beta$ -functions for the renormalised coupling constants  $u_R, b_R, s_R$  are given by (again absorbing  $1/16\pi^2$ )

$$\beta_u = u_R[-\epsilon + \frac{3u_R}{2} - 4b_R + 2s_R], \quad (47)$$

$$\beta_s = s_R[-\epsilon + \frac{5u_R}{8} + 4b_R - 2s_R], \quad (48)$$

$$\beta_b = b_R[-\epsilon + \frac{u_R}{4} + 4b_R - 2s_R]. \quad (49)$$

The FPs, as usual, are given by the zeros of the  $\beta$ -functions (47-49). Notice that  $s_R$  and  $u_R$  must have the same sign and  $b_R$  can be of any sign. The FPs are given by

- FPI: Gaussian FP -  $u_R = 0, s_R = 0, b_R = 0$ .
- FPII: DP FP -  $u_R = \frac{2\epsilon}{3}, s_R = 0, b_R = 0$ .
- FPIII:  $u_R = 0, s_R = 0, b_R = \epsilon/4$ .
- FPIV:  $s_R = 0, u_R = 8\epsilon/7, b_R = 5\epsilon/28$ .
- FPV:  $b_R = 0, u_R = \frac{16\epsilon}{17}, s_R = -\frac{7\epsilon}{34}$ . This is unphysical, since  $s_R$  and  $u_R$  have different signs. Therefore, we discard this.

We now analyse the stability of the above FPs by finding the eigenvalues  $\Lambda$  of the stability matrix corresponding to each physically meaningful FP. We find

- FPI (Gaussian FP): The eigenvalues are  $\Lambda = -\epsilon, -\epsilon, -\epsilon$ . The negativity of the all the eigenvalues indicate that this FP is unstable in all directions.
- FPII (DP FP): The eigenvalues are  $\Lambda = \epsilon, \frac{-5\epsilon}{6}, \frac{-7\epsilon}{12}$ . Thus, it is only stable along the  $u_R$ -axis and unstable in the other directions at this coupling constant space.
- FPIII The eigenvalues are  $\Lambda = -2\epsilon, \epsilon, 0$ . Thus, FPIII is unstable along the  $u_R$ -direction.
- FPIV: We find that this is stable along the  $s_R$ -axis and stable and oscillating along the  $u_R - b_R$  plane in the space of renormalised coupling constants  $u_R, b_R, s_R$ : the eigenvalues  $\Lambda$  for the corresponding stability matrix are given by

$$\Lambda = \frac{17\epsilon}{14} + i\frac{5.56\epsilon}{14}, \frac{17\epsilon}{14} - i\frac{5.56\epsilon}{14}, \frac{3\epsilon}{7}, \quad (50)$$

showing positivity of the real parts of the eigenvalues (hence stable). This leaves us with FPIV as the only stable FP for Case I.

We now obtain the corresponding critical exponents. To find out the critical exponents corresponding to these fixed points we need to evaluate the Wilson's flow functions which are defined as

$$\zeta_\phi = \mu \frac{\partial}{\partial \mu} \ln Z_\phi, \zeta_{\hat{\phi}} = \mu \frac{\partial}{\partial \mu} \ln Z_{\hat{\phi}}, \zeta_D = \mu \frac{\partial}{\partial \mu} \ln Z_D, \zeta_\tau = \mu \frac{\partial}{\partial \mu} \ln Z_\tau - 2. \quad (51)$$

From the flow functions in Eq. (51), the critical exponents of the model in terms of the renormalised coupling constants can be obtained as shown below.

$$\eta_\phi = \eta_{\hat{\phi}} = -\zeta_\phi \quad (52)$$

$$\frac{1}{\nu} = -\zeta_\tau \quad (53)$$

$$z_\phi = 2 - \zeta_D. \quad (54)$$

We find

- FPII (DP FP):  $\eta_\phi = \eta_{\hat{\phi}} = \frac{\epsilon}{12}, \nu^{-1} = 2 + \frac{\epsilon}{4}$ , dynamic exponent  $z_\phi = 2 - \frac{\epsilon}{12}$ .
- FPIII: Dynamic exponent  $z_\phi = 2 - \zeta_D = 2$ . Evidently, this is in contradiction with the expected weak dynamic scaling for  $\theta_R \rightarrow 0$  (see discussions above).

- FPIV:  $z_\phi = 2 - \zeta_D = 2 - u_R/8 = 2 - \epsilon/7 < 2$ . As we shall see below, for  $z_\phi < 2$ ,  $z_v = 2$ , which implies weak dynamic scaling, consistent with  $\theta_R \rightarrow 0$ . Thus, this FP represents *weak dynamic scaling*. Other critical exponents are (a) anomalous dimension  $\eta_\phi = -\zeta_\phi = u_R/8 = \epsilon/7$ , (b) inverse correlation length exponent  $1/\nu = -\zeta_\tau = 3u_R/8 + 2 = 3\epsilon/7 + 2$ .

Thus, FPIV is a stable FP in the coupling constant space spanned by  $u_R, s_R, b_R$ , that describes weak dynamic scaling at the AAPT, in accordance with our stating assumption  $\theta_R \rightarrow 0$ . In addition, note that  $\partial\beta_\theta/\partial\theta_R = u_R/8 > 0$  and  $\partial\beta_\theta/\partial\mathcal{Y}|_{\text{FPIV}} = 0$ , where  $\mathcal{Y} = u_R, b_R, s_R$ . This indicates that the weak dynamic scaling behaviour represented by FPIV is indeed stable along the  $\theta_R$  direction as well. Lastly, at the DP FP (FPII), all the critical exponents are unsurprisingly identical to their values for the usual DP problem. Since FPII is an unstable FP in the present model, due to the coupling of  $\phi$  with the environment (modeled by  $\mathbf{v}$ ), the DP FP of the original DP problem gives way to a non-DP FP that characterises the underlying AAPT.

Consider now Case II: With the scaling of  $w_R$  and  $\alpha_R$  with  $\theta_R$  for Case II, evidently the  $\beta$ -functions (39-41) are linear in the *effective coupling constants*  $u_R, m_R = w_R/\theta_R, \psi_R = \alpha_R w_R/\theta_R^2$ . The corresponding  $\beta$ -functions are obtained in a straightforward way:  $\beta_a = \mu \frac{\partial}{\partial \mu} a_R$ ,  $a = u, m, \psi$ . These  $\beta$ -functions can be used to find out the FPs present in the model by setting their values to zero. We obtain

$$Z_m = Z_w Z_\theta^{-1} = 1 + \frac{3u\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} - \frac{4m\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2}, \quad (55)$$

$$Z_\psi = Z_\alpha Z_w Z_\theta^{-2} = 1 + \frac{u\mu^{-\epsilon}}{4\epsilon} \frac{1}{16\pi^2} + \frac{2m\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2} + \frac{4\psi\mu^{-\epsilon}}{\epsilon} \frac{1}{16\pi^2}. \quad (56)$$

From these  $Z$ -factors effective  $\beta$ -functions for the corresponding renormalised coupling constants can be written down easily given by

$$\beta_u = u_R \left[ -\epsilon + \frac{3u_R}{2} - 14m_R - 4\psi_R \right], \quad (57)$$

$$\beta_m = m_R \left[ -\epsilon + \frac{3u_R}{4} - 4m_R \right], \quad (58)$$

$$\beta_\psi = \psi_R \left[ -\epsilon + \frac{u_R}{4} + 2m_R + 4\psi_R \right], \quad (59)$$

where factors of  $1/16\pi^2$  have been absorbed in the definitions of the renormalised coupling constants. The zeros of the  $\beta$ -functions (57-59) yield a number of FPs. We give the details below. Excluding the Gaussian FP, we have

- FPV: Set  $m_R = \psi_R = 0, u_R \neq 0$ , giving us the usual DP FP -  $u_R = \frac{2\epsilon}{3}$ .
- FPVI: Consider  $u_R = 0, m_R \neq 0, \psi_R \neq 0$ . We find  $m_R = -\frac{\epsilon}{4}, \psi_R = \frac{3\epsilon}{8}$ . Now,  $m = w/\theta$ ,  $m_R = w_R/\theta_R$  cannot be negative, since both  $w_R, \theta_R$  are non-negative. Thus, this FP is unphysical.
- FPIV: Now consider  $u_R \neq 0, m_R = 0, \psi_R \neq 0$ . The fixed points obtained are  $u_R = \frac{8\epsilon}{7}, m_R = 0, \psi_R = \frac{5\epsilon}{28}$ .
- FPIV: Next consider  $\psi_R = 0, u_R \neq 0, m_R \neq 0$ . We obtain  $u_R = \frac{20\epsilon}{9}, m_R = \frac{\epsilon}{6}$ .
- FPIX: Lastly, we obtain another FP  $u_R = 2\epsilon, m_R = \frac{\epsilon}{8}$  and  $\psi_R = \frac{\epsilon}{16}$ , when all the effective coupling constants are non-zero at the FP.

Having derived all the relevant FPs from the  $\beta$ -functions, we analyse the stability of these FPs by finding the eigenvalues of the stability matrix corresponding to each physically meaningful FP. We find

- FPV (DP FP): the eigenvalues of the stability matrix are  $\Lambda = \epsilon, -\frac{\epsilon}{2}, -\frac{5\epsilon}{6}$ . The positivity of the eigenvalue along the  $u_R$ -direction indicates that it is stable along the  $u_R$  axis but it is unstable along the  $m_R$  and the  $\psi_R$  axes.
- FPIV: the eigenvalues of the stability matrix are  $\Lambda = \frac{10\epsilon}{7}, \epsilon$  along the  $u_R - \psi_R$  plane and  $\Lambda = -\frac{\epsilon}{7}$  along the  $m_R$  axis. This shows that the FP is stable along the  $u_R - \psi_R$  plane but unstable as expected along the  $m_R$  direction.
- FPIV: the eigenvalues of the stability matrix are  $\Lambda = \epsilon, \frac{5\epsilon}{3}, -\frac{\epsilon}{9}$ . This shows that FP is stable in the  $u_R - m_R$  plane but are unstable along the  $\psi$  axis.
- FPIX: the eigenvalue equation yields  $\Lambda = 1.5752\epsilon, 0.059\epsilon, 1.115\epsilon$ . As all the eigenvalues are positive, this FP is stable in the whole  $u_R - m_R - \psi_R$  parameter space.

Therefore, we find that the nontrivial FP characterised by non-zero  $u_R, m_R, \psi_R$  is stable in *all* three directions in the space of the three coupling constants. Nonzero  $m_R$  and  $\psi_R$  at the FP suggest nonzero  $w_R$  and  $\alpha_R$  at the FP, indicating their relevance in a DRG sense. Thus, overall, both the environment and the feedback on it are relevant in determining the macroscopic scaling at the AAPT. Furthermore, our analyses above are limited only to the case  $\theta \rightarrow 0$ . Notice that with the obtained values  $u_R = 2\epsilon, m_R = \epsilon/8, \psi_R = \epsilon/16$  at the nontrivial FP,  $\partial\beta_\theta/\partial\theta = 0$ , making  $\theta_R = 0$  *marginal* at the FP. Thus, we are unable to comment whether  $\theta_R = 0$  is a stable FP or not, although its instability cannot be ruled out on any general ground.

To find out the critical exponents corresponding to these fixed points we need to evaluate the Wilson's flow functions as defined in Eqs. (51) above. Using the definitions of the critical exponents in terms of the flow functions, we obtain

- FPV or the DP FP  $(\frac{2\epsilon}{3}, 0, 0)$ :  
 $\eta_\phi = \eta_{\hat{\phi}} = \frac{\epsilon}{12}, \nu^{-1} = 2 + \frac{\epsilon}{4}, z_\phi = 2 - \frac{\epsilon}{12}.$
- FPIV  $(\frac{8\epsilon}{7}, 0, \frac{5\epsilon}{28})$ :  
 $\eta_\phi = \eta_{\hat{\phi}} = \frac{\epsilon}{7}, \nu^{-1} = 2 + \frac{3\epsilon}{7}, z_\phi = 2 - \frac{\epsilon}{7}.$
- FPIV  $(\frac{20\epsilon}{9}, \frac{\epsilon}{6}, 0)$ :  
 $\eta_\phi = \eta_{\hat{\phi}} = \frac{\epsilon}{9}, \nu^{-1} = 2 + \frac{\epsilon}{2}, z_\phi = 2 + \frac{\epsilon}{18}.$  Thus,  $z_\phi > 2.$
- FPIX  $(2\epsilon, \frac{\epsilon}{8}, \frac{\epsilon}{16})$ :  
 $\eta_\phi = \eta_{\hat{\phi}} = \frac{\epsilon}{8}, \nu^{-1} = 2 + \frac{\epsilon}{2}, z_\phi = 2.$

Notice that at FPs, FPIV and FPIX, where the effects of the environment and the feedback on it are relevant in a DRG sense,  $z_\phi > 2$  and  $z_\phi = 2$ , respectively. As we see in the next Section, for  $z_\phi \geq 2$ , the dynamic exponent for  $v_i$ ,  $z_v = z_\phi$ , indicating strong dynamic scaling at these FPs. This is in contradiction with the expected weak dynamic scaling at  $\theta_R = 0$ ; in other words  $z_\phi < z_v$  is expected. Therefore, FPIV and FPIX are unphysical FPs. In contrast at FPIV,  $z_\phi < 2$ ; with  $z_v = 2$  (see below) this corresponds to weak dynamic scaling. However, it is unstable along the  $m_R$ -direction. Hence, we find that there is only one FP (FPV) that is stable in all directions and describe weak dynamic scaling for the AAPT, and so represents a physically correct scaling behaviour at the AAPT.

## B. Scaling exponents of the broken symmetry field

To obtain the scaling exponents of the broken symmetry field  $\mathbf{v}$ , we start from Eq. (8) for  $v_i$ . Evidently, if  $\chi = 0$ , i.e., if the dynamics of  $v_i$  is autonomous,  $v_i$  can be solved exactly with

$$\langle v_i(\mathbf{k}, \omega) v_j(-\mathbf{k}, -\omega) \rangle = \frac{2D_0\delta_{ij}}{\omega^2 + \lambda^2 k^4}, \quad \langle v_i(\mathbf{k}, t) v_j(-\mathbf{k}, t) \rangle = \frac{D_0}{\lambda k^2}, \quad (60)$$

and hence the exponents of  $v_i$  are also known exactly: Dynamic exponent  $z_v = 2$  and anomalous dimension  $\eta_v = 0$ . When  $\chi \neq 0$ , one can still obtain an exact closed form for the correlator of  $v_i$ , owing to the linearity of the feedback term in Eq. (8):

$$\langle v_i(\mathbf{k}, \omega) v_j(-\mathbf{k}, -\omega) \rangle = \frac{2D_0\delta_{ij}}{\omega^2 + \lambda^2 k^4} + \frac{\chi^2 \langle |\phi(\mathbf{k}, \omega)|^2 \rangle k_i k_j}{\omega^2 + \lambda^2 k^4}. \quad (61)$$

Noting that in terms of the scaling exponents and in terms of the renormalised parameters

$$\langle |\phi(\mathbf{k}, \omega)|^2 \rangle \sim \frac{1}{k^{2-\eta_\phi}} \frac{D_R k^{z_\phi}}{\omega^2 + D_R^2 k^{2z_\phi}} \quad (62)$$

leading to

$$\langle v_i(\mathbf{k}, t) v_j(-\mathbf{k}, 0) \rangle \sim \frac{\chi^2 k_i k_j D_R k^{z_\phi}}{k^{2-\eta_\phi}} \left[ \frac{\exp(-D_R k^{z_\phi} t)}{2D_R k^{z_\phi} (\lambda^2 k^4 - D_R^2 k^{2z_\phi})} + \frac{\exp(-\lambda k^2 t)}{2\lambda k^2 (D_R^2 k^{2z_\phi} - \lambda^2 k^4)} \right] + \frac{D_0 \exp(-\lambda k^2 t) \delta_{ij}}{2\lambda k^2}, \quad (63)$$

yielding in the hydrodynamic limit  $k \rightarrow 0$  and, assuming  $z_\phi < 2$ , for large time  $t \gg 1/(D k^{z_\phi})$ ,

$$\langle v_i(\mathbf{k}, t) v_j(-\mathbf{k}, 0) \rangle \sim \frac{\exp(-\lambda k^2 t) \delta_{ij}}{2\lambda k^2} + \frac{\chi^2 k_i k_j D_R k^{z_\phi}}{k^{2-\eta_\phi}} \frac{\exp(-\lambda k^2 t)}{2\lambda k^2 [D_R^2 k^{2z_\phi} - \lambda^2 k^4]}, \quad (64)$$

giving a dynamic exponent  $z_v = 2$ , and hence weak dynamic scaling. On the other hand for  $z_\phi = 2$ , one, of course, has  $z_v = 2$ , displaying strong dynamic scaling. Furthermore, in the event  $z_\phi > 2$ , it is clear from the preceding discussion that  $z_\phi = z_v$ , indicating strong dynamic scaling. We can further obtain results for the anomalous dimension  $\eta_v$  of  $v_i$ : We have for the equal-time correlator

$$\langle v_i(\mathbf{k}, t) v_j(-\mathbf{k}, t) \rangle \sim \frac{D_0 \delta_{ij}}{\lambda k^2} + \frac{\chi^2 k_i k_j}{k^{2-\eta_\phi}} \frac{1}{D_R k^{z_\phi}}. \quad (65)$$

Thus, if  $D_0 = 0$  then  $\eta_v = \eta_\phi - z_\phi - 2$ . However, if  $D_0 \neq 0$ , then if  $-\eta_\phi + z_\phi - 2 < 0$  then the first term on the right hand side of (65) dominates, giving  $\eta_v = 0$ , else  $\eta_v = \eta_\phi - z_\phi + 2$ . Thus, at the nontrivial FP (FPVIII),  $\eta_v = \frac{2\epsilon}{7}$ . This completes the discussions on the enumeration of the scaling exponents of  $\mathbf{v}$ .

#### IV. SUMMARY AND OUTLOOK

In this article, we have constructed a simple model and studied it to find how the mutual interactions between a density undergoing an AAPT and its surrounding fluctuating environment affect the universal scaling properties of both the density field and the environment at the extinction transition. We have used a broken symmetry mode, modeled by a vector field, to represent the fluctuating environment. We have used a perturbative (up to the one-loop order) DRG calculation to extract the relevant scaling exponents that define the AAPT. The zeros of the DRG  $\beta$ -functions yield DRG fixed points, each representing a phase, characterised by a set of values for the scaling exponents. Due to the algebraic complications involved, we have been able to obtain the FPs only in the limit  $\theta_R \rightarrow 0$ . This corresponds to weak dynamic scaling with  $z_\phi < z_v$ . Thus, the possibility of  $z_\phi > z_v$  is effectively ignored. We obtain one FP (FPIV above) with  $\theta_R \rightarrow 0$  that yields physically acceptable results for the scaling exponents at the AAPT. This FP is stable in all the directions in the space of the effective coupling constants, and also in the direction of  $\theta_R$ . Thus, we speculate that our model displays only weak dynamic scaling with the scaling properties at the AAPT being given by FPIV. The quantitative accuracy of the scaling exponents obtained are limited by the approximations involved. Nonetheless, since the exponents at FPIV is different from their usual DP counterparts, we are able to show that both the environmental influence and the feedback on it by the density undergoing the AAPT are generally relevant in a DRG sense. While we speculate about our model displaying only weak dynamic scaling, existence of stable FPs with finite  $\theta_R$  corresponding to strong dynamic scaling (i.e.,  $z_\phi = z_v$ ) should be investigated numerically from the zeros of the  $\beta$ -functions (34-37) with finite  $\theta_R$  as complementary to the present study. We briefly discuss the possibility of FPs in Appendix A with  $\theta_R \rightarrow \infty$  (i.e., with  $z_\phi > z_v$ ). We show that there are no stable FPs there. Regardless of the limitations of our calculations here, generally at the physically acceptable stable FP obtained here, at which the feedback is relevant, not only the scaling exponents of the density field undergoing AAPT are affected by the environment, even the scaling exponents of the coupled broken symmetry mode (the environment) should be affected in turn. This clearly establishes the relevance of feedback (in a DRG sense) for both the density and the broken symmetry fields.

As discussed above, our model equation (8) and its symmetry (polar symmetry) are simplified versions of realistic models. More realistic models include, e.g., the equations of motion of polar order parameter of an active (nonequilibrium) polar system, which couples to the concentration of the active particles in a way similar to Eq. (8) above [20]. However, the structure of the polar order parameter equation in an active system is much more complicated [20], than the simplified equation for  $\mathbf{v}$  that we have used here. It would be interesting to investigate how different symmetries of  $\mathbf{v}$  (e.g., polar versus nematic) may change the emerging scaling behaviour at the critical point. Our work highlights the importance of feedback of the density undergoing AAPT on the environmental dynamics. However, our study here is confined to illustrating the effects of linear feedback. This would be relevant, e.g., in a bacteria colony in its ordered state undergoing birth and death. There may, however, be situations where the feedback is nonlinear. An interesting example could be the AAPT of a density field being advected by an incompressible velocity field [25] or the birth-growth of bacteria in their nematic ordered state. It will be theoretically interesting to study the effects of nonlinear feedback, especially in the context of weak and strong dynamic scaling. Our work should also be useful in understanding other realistic situations, e.g., extinction transition in an orientationally ordered bacteria film resting on a fluctuating surface or a fluctuating membrane.

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## Appendix A: Large $\theta_R$ limit

We here consider the stability of the DP FP in the limit  $\theta_R \rightarrow \infty$  (equivalently, bare  $\theta \rightarrow \infty$ ). In this limit, we should get  $z_\phi > z_v$  (again weak dynamic scaling, but the opposite of FPIV). Let  $\bar{\theta} = 1/\theta$ , so that  $\bar{\theta} \rightarrow 0$ . In this limit, no divergences are encountered in the  $Z$ -factors (31-32), or, in the  $\beta$ -functions (34-37) and the limit  $\bar{\theta} \rightarrow 0$  may be taken directly and smoothly. The relevant  $\beta$ -functions for the renormalised coupling constants  $u_R$ ,  $w_R$ ,  $\alpha_R$  in this limit are

$$\beta_u = u_R[-\epsilon + \frac{3u_R}{2}], \quad (A1)$$

$$\beta_w = w_R[-\epsilon + \frac{7u_R}{8}], \quad (A2)$$

$$\beta_\alpha = \alpha_R[-\frac{3u_R}{8}], \quad (A3)$$

$$\beta_{\bar{\theta}} = \bar{\theta}_R[-\frac{u_R}{8}]. \quad (A4)$$

Thus, apart from the trivial Gaussian FP, the only other FP is  $u_R = 2\epsilon/3, w_R = 0, \alpha_R = 0$  (DP FP) together with  $\bar{\theta}_R = 0$ . It is not surprising that with  $\bar{\theta}_R = 0$ , DP FP is the only non-zero FP left in the system, since with (assumed)  $z_\phi > z_v$   $\mathbf{v}$ -fluctuations vanish for time-scales  $t \gg 1/(\lambda k^{z_v})$ . However, this FP is *unstable* in all the three directions of  $w_R, \alpha_R, \bar{\theta}_R$ . Again, this is consistent with our argument in Sec. IIIB that  $z_\phi \leq z_v$  necessarily, precluding the possibility of  $z_\phi > z_v$ , as it would be for  $\bar{\theta}_R \rightarrow 0$ . Our analysis here however does not rule out the possibility of FPs with non-zero but finite  $\theta_R$  (i.e., with strong dynamic scaling  $z_v = z_\phi$ ). Such FPs, if exist, may be analysed by numerically solving for the zeros of the  $\beta$ -functions (34-37). We do not do this here.

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